الجامعة السورية الخاصة
SYRIAN PRIVATE UNIVERSITY

كـلـيـــة هندسة الحاسوب والمعلوماتية
Computer and Informatics Engineering Faculty

Electric Circuits I

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Chapter 8
Second-Order Circuits

8.1 Examples of 2nd order RCL circuit

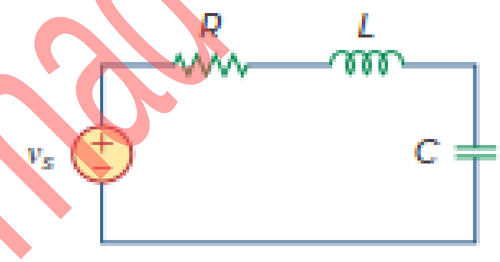
- A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.
- Typical examples of second-order circuits:

  (a) series RLC circuit,

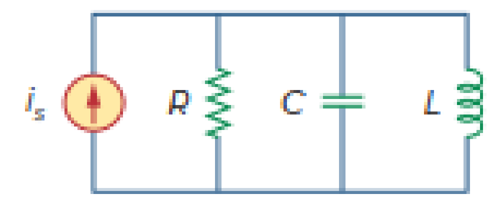
  (b) parallel RLC circuit,

  (c) RL circuit,

  (d) RC circuit.



(a)



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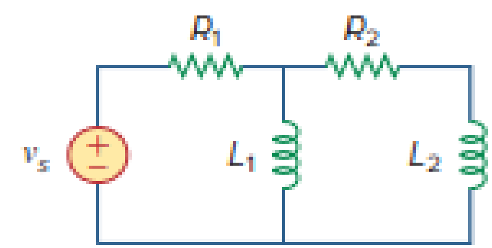

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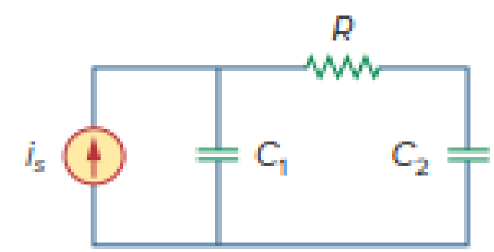


(d)

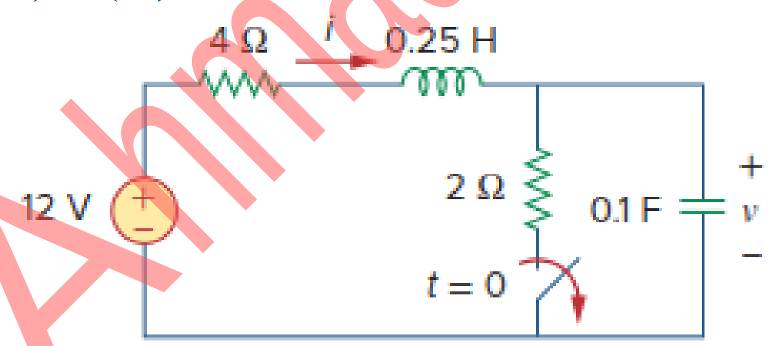
8.2 Finding Initial and Final Values

- The major problem in analysis of the second-order circuits is finding the initial and final conditions on circuit variables: $v(0)$, $i(0)$, $dv(0)/dt$, $di(0)/dt$, $i(\infty)$, and $v(\infty)$.
- There are **two key points** to keep in mind in determining the initial conditions.
  - First—we must carefully handle the polarity of voltage $v(t)$ across the capacitor and the direction of the current $i(t)$ through the inductor.
  - Second, keep in mind that the capacitor voltage is always *continuous* so that $v(0^+) = v(0^-)$

and the inductor current is always *continuous* so that $i(0^+) = i(0^-)$

where $t = 0^-$ denotes the time just **before** a switching event and $t = 0^+$ is the time just **after** the switching event, assuming that the switching event takes place at $\boldsymbol{t = 0}$.

Example 8.1. The switch in Fig. has been closed for a long time. It is open at $t = 0$. Find: $i(0^+)$, $v(0^+)$, $di(0^+)/dt$, $dv(0^+)/dt$, $i(\infty)$, $v(\infty)$.



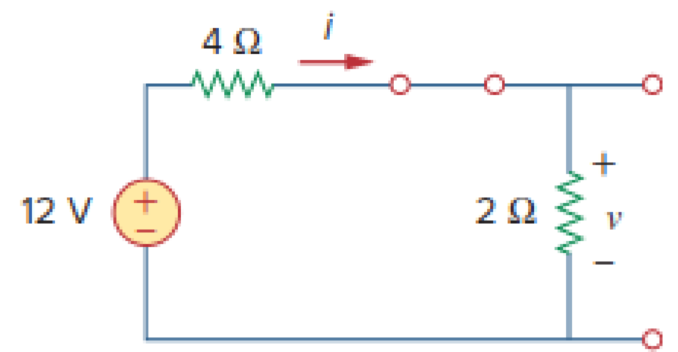
Solution:

a) If the switch is closed a long time before $t = 0$, it means that the circuit has reached dc steady state at $t = 0$.
At dc steady state, the inductor acts like a short circuit, while the capacitor acts like an open circuit, so we have the circuit in Fig.(a) at $t = 0-$.
Thus,

$$i(0^-) = \frac{12}{4+2} = 2\,\text{A}, \quad v(0^-) = 2i(0^-) = 4\,\text{V}$$

As the inductor current and the capacitor voltage cannot change abruptly,

$$i(0^+) = i(0^-) = 2\,\text{A}, \quad v(0^+) = v(0^-) = 4\,\text{V}$$

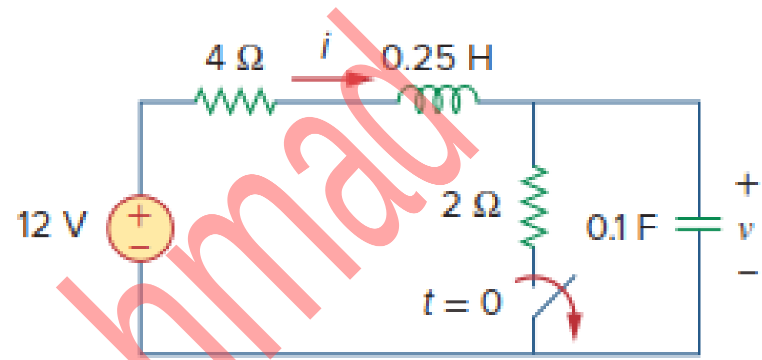


(a)

b) At $t = 0+$, the switch is open; Fig.(b).

The same current flows through both the inductor and capacitor.

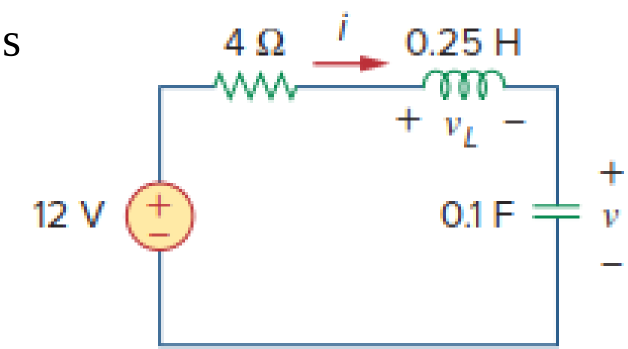
Hence,



$$i_C(0^+) = i(0^+) = 2\,\text{A}$$

$$i_C = C\frac{dv}{dt} \Leftrightarrow \frac{dv}{dt} = \frac{i_C}{C} \Rightarrow \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\ \text{V/s}$$

Similarly, $v_L = L\frac{di}{dt} \Leftrightarrow \frac{di}{dt} = \frac{v_L}{L}$



(b)

Applying KVL to the loop:

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

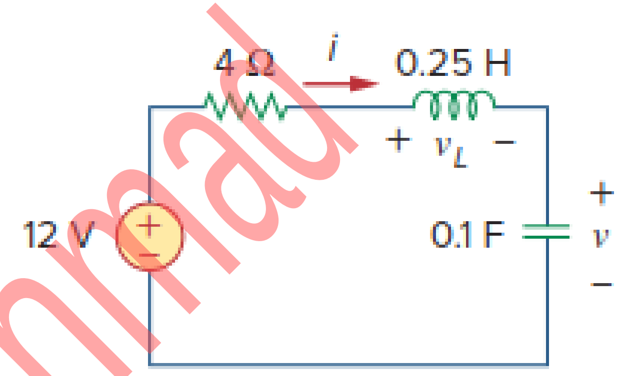
$$\Rightarrow v_L(0^+) = 12 - 8 - 4 = 0$$

$$\Rightarrow \frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\ \text{A/s}$$

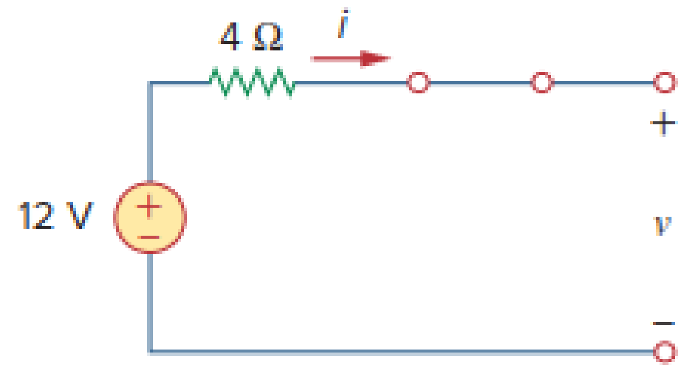
c) For $t > 0$, the circuit undergoes transience. But as $t \to \infty$, the circuit reaches steady state again.

The inductor acts like a short circuit and the capacitor like an open circuit, so that the circuit in Fig.(b) becomes that shown in Fig. (c), from which we have

$$i(\infty) = 0\,\text{A}, \quad v(\infty) = 12\,\text{V}$$

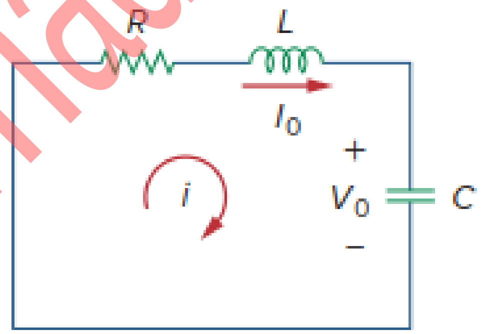


(b)



(c)

8.3 The Source-Free Series *RLC* Circuits



- The solution of the source-free series *RLC* circuit is called as the **natural response** of the circuit.
- The circuit is excited by the energy initially stored in the capacitor (V_0) and inductor (I_0).
  - Thus, at $t = 0$,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0, \qquad i(0) = I_0$$

  - Applying KVL around the loop:

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{0} i(\tau)\,d\tau = 0$$

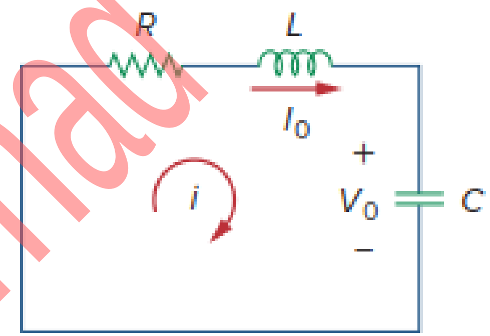
  - By differentiate with respect to *t*, we get

$$\boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0}$$

  - This is a **second-order differential equation.**

- The **initial value *i*** is:

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0 \Leftrightarrow \frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$



- For the first-order circuit, the solution for i is of exponential form: $i = Ae^{st}$
  where A and s are constants to be determined.
- Thus, $$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \Rightarrow As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

$$\Rightarrow Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

- Since $i = Ae^{st}$ is the assumed solution, so only $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$

  with characteristic roots $$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

  or $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$ $\quad \alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$ → $s^2 + 2\alpha s + \omega_0^2 = 0$

- The two values of ***s*** indicate that there are two possible solutions for *i*,

$$i_1 = A_1 e^{s_1 t}, \quad i_2 = A_2 e^{s_2 t} \Rightarrow i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

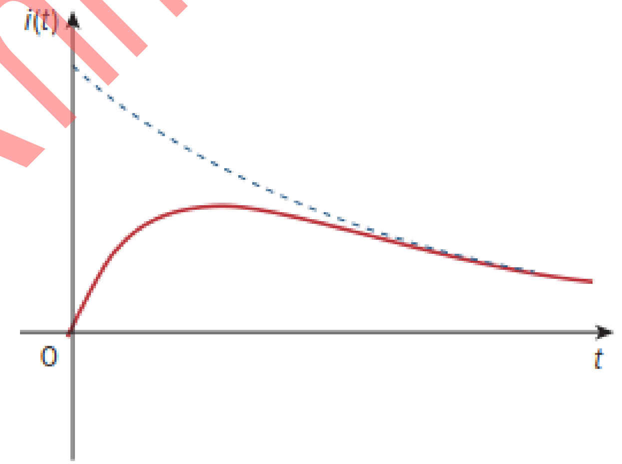
where the constants A_1 and A_2 are determined from the initial values $i(0)$ and $di(0)/dt$.

There are **three types of solutions:**

1. Overdamped Case (فوق متضائل),

$$\alpha > \omega_0, \text{ i.e. } C > 4L/R^2 \Rightarrow i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

– A typical overdamped response in Fig.(a).



(a)

2. Critically Damped Case (متضائل بشكل حرج)

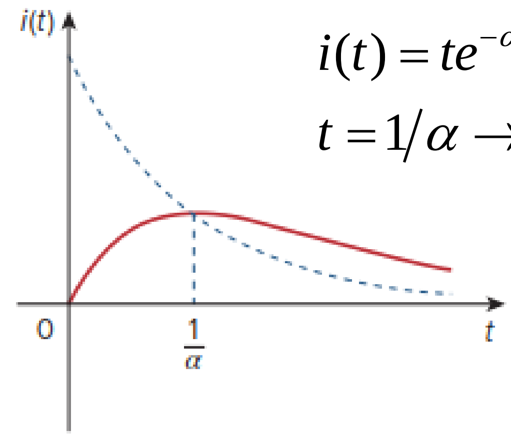
$$\alpha = \omega_0, \text{ i.e. } C = 4L/R^2 \Rightarrow s_1 = s_2 = -\alpha = -\frac{R}{2L}$$

$$\Rightarrow i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = (A_1 + A_2)e^{-\alpha t} = A_3 e^{-\alpha t}$$

A typical critically damped response Fig.(b).

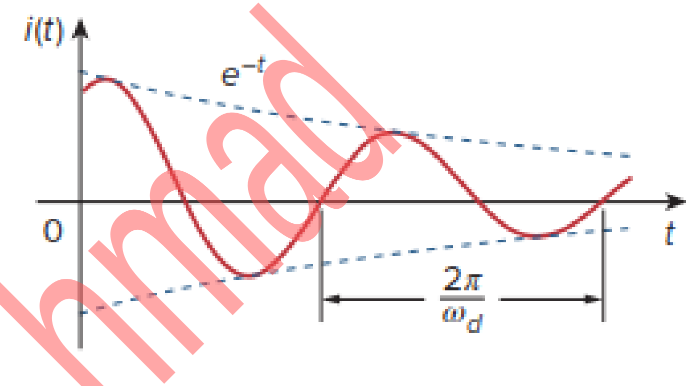
$$i(t) = te^{-\alpha t}$$

$$t = 1/\alpha \rightarrow \text{max}$$



(b)

3. Underdamped Case (تحت متضائلة)



(c)

$$\alpha < \omega_0, \text{ i.e. } \; C < 4L/R^2$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha \pm j\omega_d$$

$$j = \sqrt{-1}, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

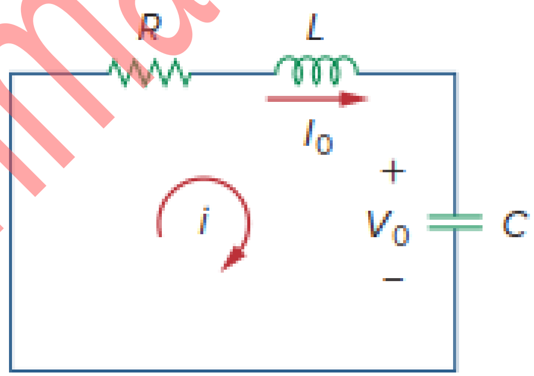
- $\boldsymbol{\omega_0}$ is often called the undamped natural frequency,
- ω_d is called the damped natural frequency.

- The **natural response** is

$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$
$$B_1 = A_1 + A_2, \quad B_2 = j(A_1 - A_2)$$

- For simply, $i(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$
- The response has a *time constant* of **1/*α*** and a *period* of ***T* = 2*π*/$\boldsymbol{\omega_d}$**.

Example 8.2. In Fig., R = 40 Ω, L = 4 H, and C = 1⁄4 F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?



Solution,

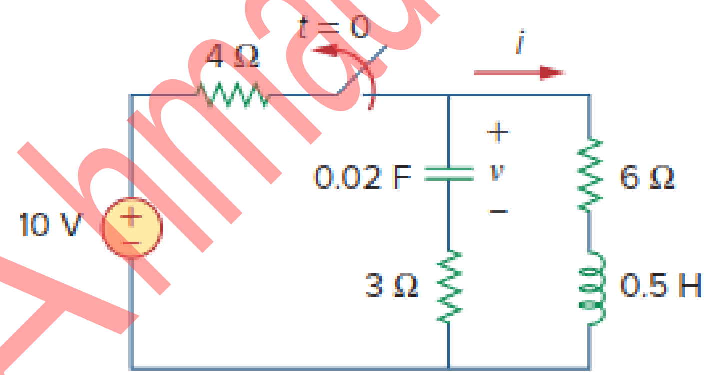
$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

$$s_1 = -0.101, \quad s_2 = -9.899$$

Since $\alpha > \omega_0$, we conclude that the response is overdamped.

Example 8.3. Find $i(t)$ in the circuit of Fig. Assume that the circuit has reached steady state at $t = 0^-$.



Solution,

For $t < 0$, the switch is closed.

capacitor ➔ open circuit,

inductor ➔a shunted circuit.

The equivalent circuit, Fig.(a).

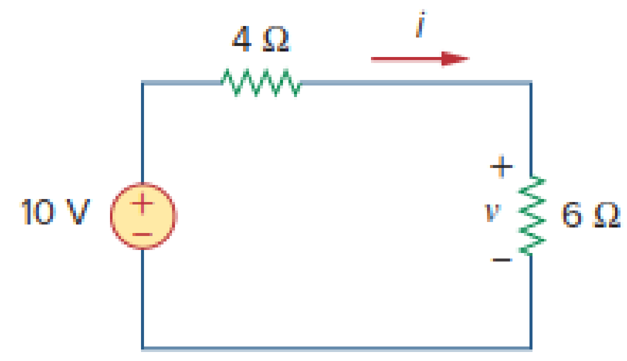
Thus, at $t = 0$,

- the **initial current** through the inductor is

$$i(0) = \frac{10}{4+6} = 1\,\text{A}$$

- the **initial voltage** across the capacitor

$$v(0) = 6i(0) = 6\,\text{V}$$



(a)

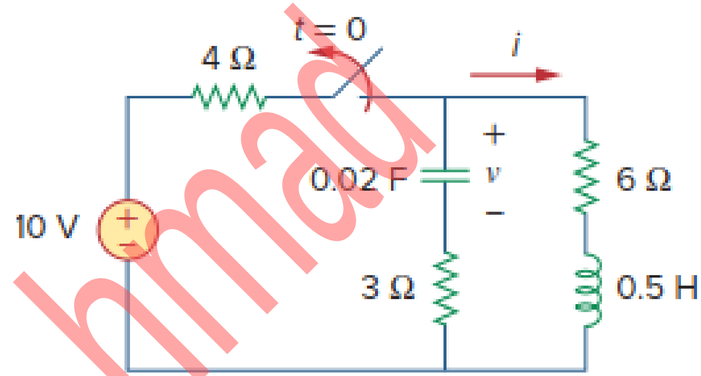
For ***t* > 0**, the switch is opened.

The voltage source is disconnected, Fig.(b).

(source free series RLC circuit),

where the 3-Ω and 6-Ω resistors are in series

when the switch is opened: $R = 3 + 6 = 9\Omega$



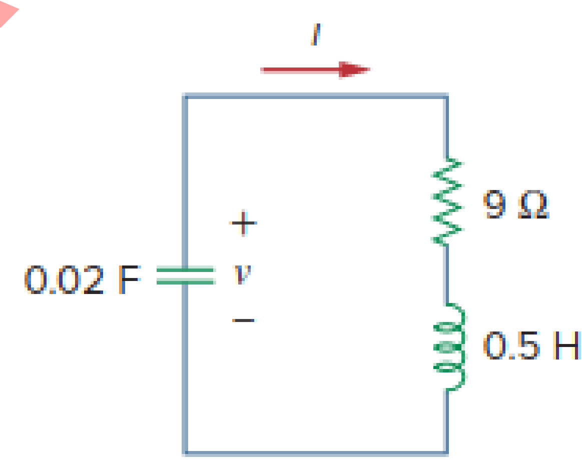
The **roots** are calculated as follows:

$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100}$$

$$= -9 \pm j4.359 = -\alpha \pm j\omega_d$$

$$\Rightarrow \alpha = 9, \ \omega_d = 4.359$$



(b)

Hence, the response is underdamped ($\alpha < \omega_0$); that is,

$$i(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$

At $t = 0$, $i(0) = 1\text{A} = A_1$

But, $$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}(RI_0 + V_0) = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6\ \text{A/s}$$

Note: $V_0 = -v(0)$, because the polarity of v in Fig. (b) is opposite that in Fig. for source-free series *RLC* circuit (slid 8).

Thus,

$$\frac{di}{dt} = \frac{d}{dt}\left[e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)\right]$$

$$= -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) + e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)$$

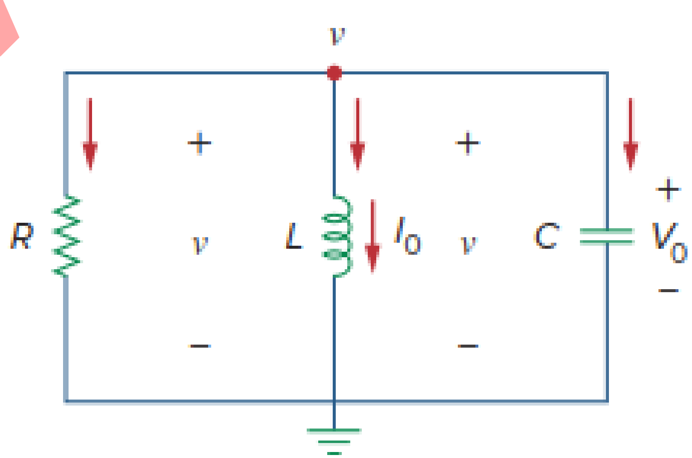
At $t = 0$, $$\left.\frac{di}{dt}\right|_{t=0} = -6 = -9(A_1 + 0) + 4.359(-0 + A_2),\ \text{for}\ A_1 = 1 \Rightarrow A_2 = 0.6882$$

Finally, $$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t)\ \text{A}$$

8.4 Source-Free Parallel *RLC* Circuits

- Parallel *RLC* circuits find many practical applications, notably in *communications networks and filter designs*.
- Consider the parallel *RLC* circuit shown in Fig.
  - Assume initial inductor current I_0 and initial capacitor voltage V_0,

$$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)dt, \quad v(0) = V_0$$



- Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{0} v(\tau)d\tau + C\frac{dv}{dt} = 0$$

- Taking the *derivative* with respect to *t* and *dividing* by *C* results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \Leftrightarrow s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

where *s* is first derivative, s^2 is second derivative.

- The roots of characteristic equation are

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \Rightarrow s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$ $$\alpha = \frac{1}{2RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

There are **three possible solutions**,

1. Overdamped Case.

$$\alpha > \omega_0, \text{ i.e. } L > 4R^2C \Rightarrow$$

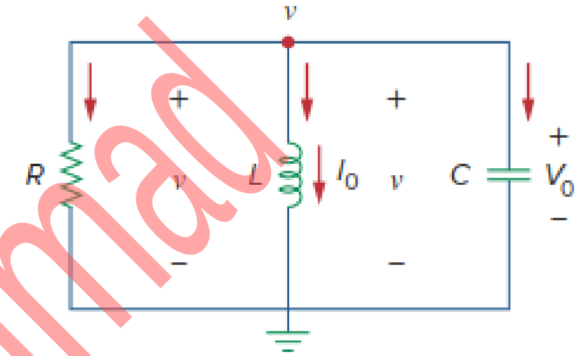
The response is $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

2. Critically Damped Case.

$$\alpha = \omega_0, \text{ i.e. } L = 4R^2C \Rightarrow$$

The response is $v(t) = (A_1 + A_2 t)e^{-\alpha t}$, $s_1 = s_2 = -\alpha$ (real)

3. Underdamped Case.



$$\alpha < \omega_0, \text{ i.e. } \; L < 4R^2C \Rightarrow$$

The response is

$$v(t) = \left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)e^{-\alpha t}, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}, \quad s_{1,2} = -\alpha \pm j\omega_d$$

- The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$.

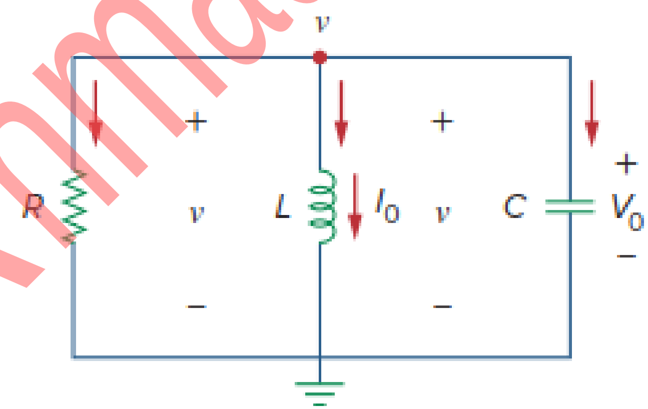
$$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)dt, \quad v(0) = V_0$$

at the top node

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{0} v(\tau)d\tau + C\frac{dv}{dt} = 0 \Leftrightarrow \frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt}$$

or $$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$$

Example 8.4. In the parallel circuit of Fig., find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF. Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.



Solution:

CASE 1: $R = 1.923\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times1.923\times10\times10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times10\times10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is overdamped. So,

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \Rightarrow s_1 = -2,\ s_2 = -50$$

the corresponding response is $v(t) = A_1 e^{-2t} + A_2 e^{-50t}$

We now apply the initial conditions to get A_1 and A_2 : $\quad v(0) = 5 = A_1 + A_2 \qquad (1)$

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \Rightarrow \frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = \frac{5+0}{1.923\times10\times10^{-3}} = -260$$

But $v(t) = A_1 e^{-2t} + A_2 e^{-50t} \Rightarrow \frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$

At $t = 0$, $\quad -260 = -2A_1 - 50A_2 \quad (2)$

From Eqs. (1) and (2), we obtain $A_1 = -0.2083$ and $A_2 = 5.208$.

Finally, $$v(t) = -0.2083e^{-2t} + 5.208e^{-50t}$$

CASE 2: $R = 5\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is damped. So,

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \Rightarrow s_1 = s_2 = -10, \quad v(t) = (A_1 + A_2 t)e^{-\alpha t} = v(t) = (A_1 + A_2 t)e^{-10t}$$

To get A_1 and A_2, we apply the initial conditions:

$$v(0) = 5 = A_1, \quad \frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = \frac{5+0}{5 \times 10 \times 10^{-3}} = -100$$

But, $\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$

At $t = 0$, $\quad -100 = -10A_1 + A_2 \Rightarrow A_2 = -50$

Finally, $$v(t) = (5 - 50t)e^{-10t} \text{ V}$$

CASE 3: $R = 6.25\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times5\times10\times10^{-3}} = 10, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times10\times10^{-3}}} = 10$$

Since $\alpha < \omega_0$ in this case, the response is underdamped. So,

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \Rightarrow s_{1,2} = -8 \pm j6 = -\alpha \pm j\omega_d \Rightarrow \omega_d = 6$$

$$v(t) = \left(A_1 \cos 6t + A_2 \sin 6t\right)e^{-8t}$$

To get A_1 and A_2 , we apply the initial conditions:

$$v(0) = 5 = A_1, \quad \frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5+0}{6.25\times10\times10^{-3}} = -80$$

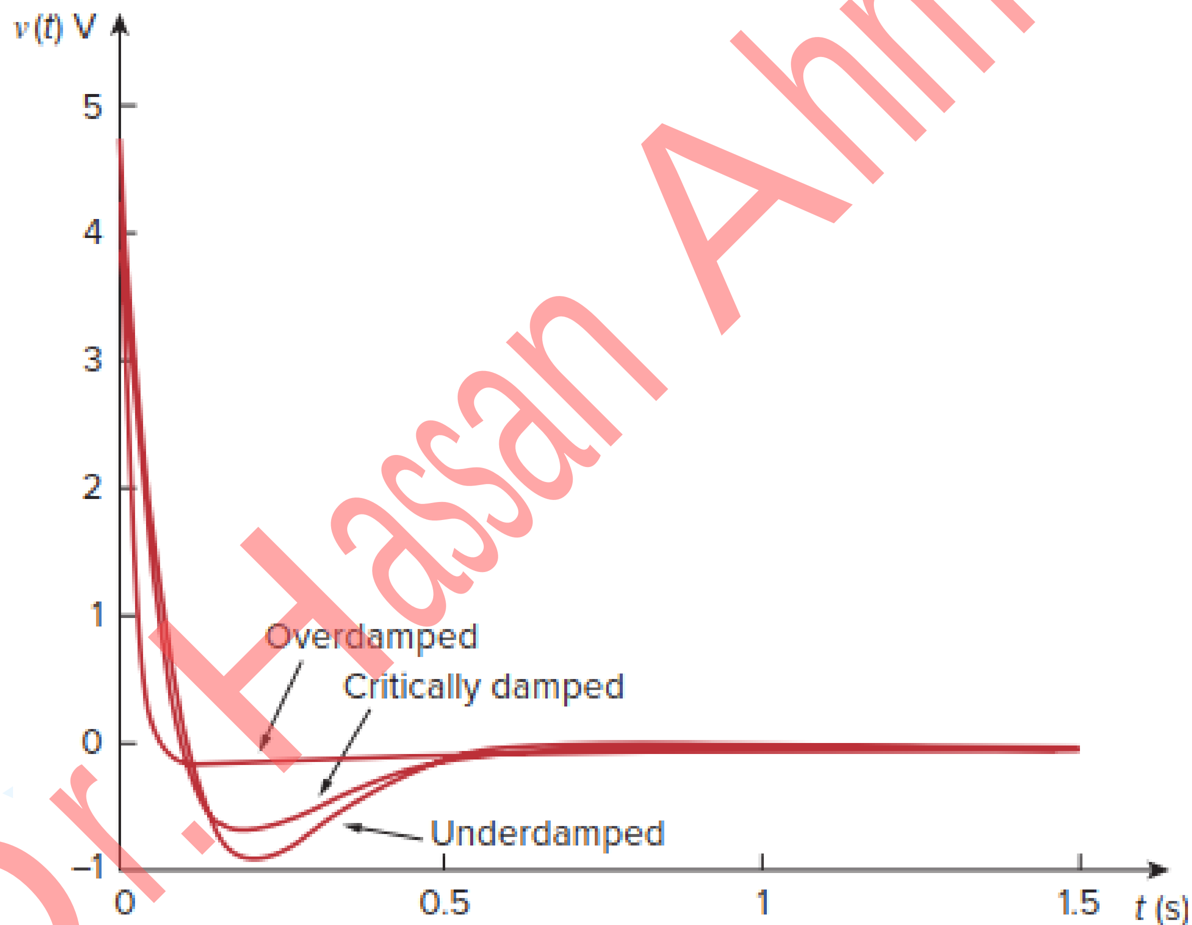
But,
$$\frac{dv}{dt} = \frac{d}{dt}\left(A_1 \cos 6t + A_2 \sin 6t\right)e^{-8t}$$

$$= (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$, $\quad -80 = -8A_1 + 6A_2 \Rightarrow A_2 = -6.667$

Finally, $\quad v(t) = \left(5\cos 6t + 6.667 \sin 6t\right)e^{-8t}$

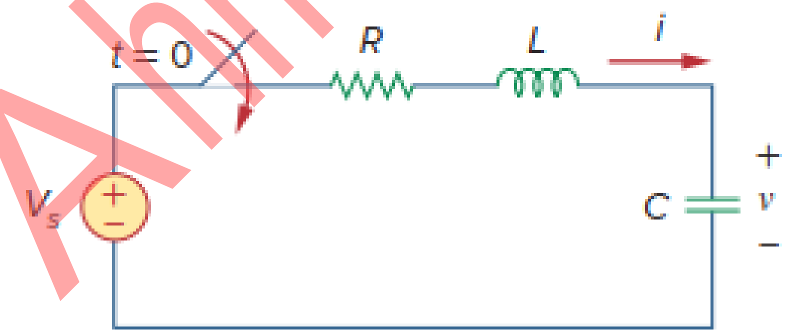
Notice that by ***increasing*** the value of *R*, the *degree of damping* ***decreases*** and the *responses differ* (تتباين = تختلف). Figure plots the three cases of Example 8.4.



8.5 Step-Response of Series *RLC* Circuits

- The **step response** is obtained by the sudden application of a dc voltage.

- Applying KVL around the loop for t > 0,



$$L\frac{di}{dt}+Ri+v=V_s, \quad i=C\frac{dv}{dt}$$

$$\Rightarrow \frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC}$$

The solution to this Eq. has two components: the *transient response* $v_t(t)$ and the *steady-state response* $v_{ss}(t)$; that is, $v(t)=v_t(t)+v_{ss}(t)$

The transient response $v_t(t)$ is the component of the total response that dies out with time.

Therefore, the **transient response** for three cases:

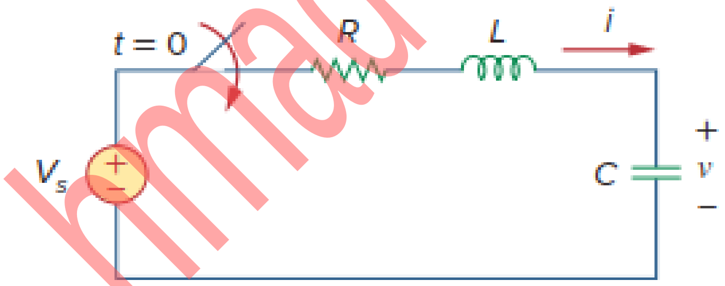
$$v(t)=A_1e^{s_1t}+A_2e^{s_2t} \quad \text{(Ovedamped)}$$

$$v(t)=(A_1+A_2t)e^{-\alpha t} \quad \text{(Critically damped)}$$

$$v(t)=\left(A_1\cos\omega_d t+A_2\sin\omega_d t\right)e^{-\alpha t} \quad \text{(Underdamped)}$$

- The **steady-state response** is the final value of *v*(*t*).

$$v_{ss}(t) = v(\infty) = V_s$$

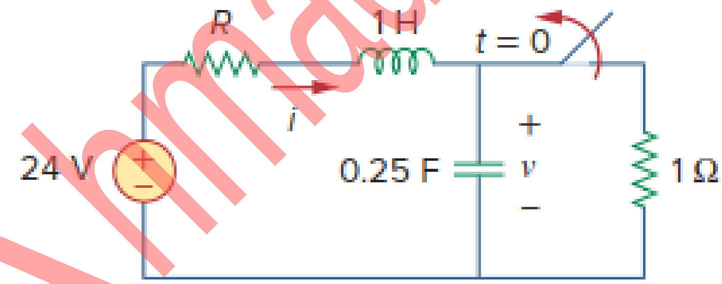


- Thus, the complete solutions are:

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Ovedamped)}$$
$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)}$$
$$v(t) = V_s + \left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)e^{-\alpha t} \quad \text{(Underdamped)}$$

- The values of the constants A_1 and A_2 are obtained from the initial conditions: $v(0)$ and $dv(0)/dt$.

Example 8.5. For the circuit in Fig., find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, and $R = 1\ \Omega$.



Solution:

CASE 1: $R = 5\ \Omega$,

For $t < 0$, the switch is closed for a long time.

capacitor ➔open circuit, inductor ➔short circuit.

$$i(0) = \frac{24}{5+1} = 4\,\text{A}, \qquad v(0) = 1 \times i(0) = 4\,\text{V}$$

For $t > 0$, the switch is opened, ➔1- Ω resistor disconnected.

The characteristic roots are determined as follows:

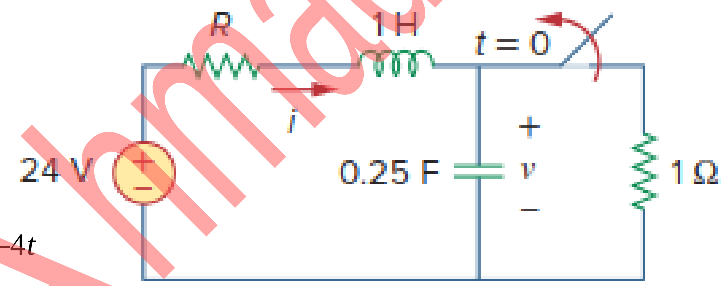
$$\alpha = \frac{R}{2L} = \frac{5}{2\times 1} = 2.5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 0.25}} = 2 \Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, -4$$

Since $\alpha > \omega_0$, we have the overdamped natural response. The total response is therefore

$$v(t) = v_{ss} + (A_1 e^{-t} + A_2 e^{-4t}) = v(t) = 24 + (A_1 e^{-t} + A_2 e^{-4t})$$

To find A_1 and A_2 using the initial conditions:

$$v(0) = 4 = 24 + A_1 + A_2 \Rightarrow A_1 + A_2 = -20 \qquad (1)$$



$$i(0) = C\frac{dv(0)}{dt} = 4 \Rightarrow \frac{dv(0)}{dt} = \frac{4}{C} = \frac{4}{0.25} = 16$$

But, $$\frac{dv}{dt} = \frac{d}{dt}\left[24 + (A_1 e^{-t} + A_2 e^{-4t})\right] = -A_1 e^{-t} - 4A_2 e^{-4t}$$

At $t = 0$, $$\frac{dv(0)}{dt} = 16 = -A_1 - 4A_2 \qquad (2)$$

From Eqs. (1) and (2): $A_1 = -64/3, \quad A_2 = 4/3$

Finally, $$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t}) \text{ V}$$

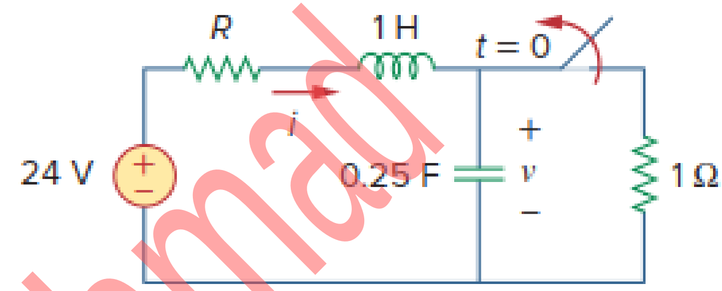
Since the inductor and capacitor are in series for $t > 0$, the inductor current is the same as the capacitor current. Hence,

$$i(t) = C\frac{dv}{dt} \Rightarrow i(t) = \frac{4}{3}(4e^{-t} - e^{-4t}) \text{ A}$$

CASE 2: $R = 4\ \Omega$,

For $t < 0$, the switch is closed for a long time.

capacitor ➔ open circuit, inductor ➔ short circuit.



$$i(0) = \frac{24}{4+1} = 4.8\,\text{A}, \qquad v(0) = 1\times i(0) = 4.8\,\text{V}$$

$$\alpha = \frac{R}{2L} = \frac{4}{2\times 1} = 2, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 0.25}} = 2 \Rightarrow s_1 = s_2 = -\alpha = -2$$

So, have the critically damped natural response. The total response is therefore

$$v(t) = 24 + (A_1 + A_2 t)e^{-2t}, \qquad v(0) = 4.8 = 24 + A_1 \Rightarrow A_1 = -19.2 \quad (1)$$

$$i(0) = C\frac{dv(0)}{dt} = 4.8 \Rightarrow \frac{dv(0)}{dt} = \frac{4.8}{C} = 19.2$$

$$\frac{dv}{dt} = \frac{d}{dt}\left[24 + (A_1 + A_2 t)e^{-2t}\right] = (-2A_1 - 2tA_2 + A_2)e^{-2t}$$

At $t = 0$, $\quad \frac{dv(0)}{dt} = 19.2 = -2A_1 + A_2 \quad (2)$

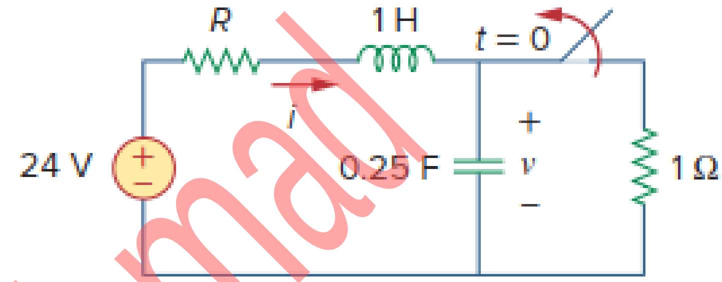
From Eqs. (1) and (2): $\quad A_1 = -19.2, \quad A_2 = -19.2$

Finally, $\quad v(t) = 24 - 19.2(1+t)e^{-2t}\ \text{V} \qquad i(t) = C\frac{dv}{dt} \Rightarrow i(t) = (4.8e^{-t} + 9.6e^{-2t})\ \text{A}$

CASE 3: $R = 1\ \Omega$,

For $t < 0$, the switch is closed for a long time.

capacitor ➔ open circuit, inductor ➔ short circuit.



$$i(0) = \frac{24}{1+1} = 12\,\text{A}, \qquad v(0) = 1 \times i(0) = 12\,\text{V}$$

$$\alpha = \frac{1}{2L} = \frac{1}{2\times 1} = 0.5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 0.25}} = 2$$

Since $\alpha < \omega_0$ in this case, the response is underdamped. So,

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \Rightarrow s_{1,2} = -0.5 \pm j1.936 = -\alpha \pm j\omega_d \Rightarrow \omega_d = 1.936$$

The total response is: $v(t) = 24 + \left(A_1 \cos 1.936t + A_2 \sin 1.936t\right) e^{-0.5t}$

$$v(0) = 12 = 24 + A_1 \Rightarrow A_1 = -12 \qquad (1)$$

$$i(0) = C\frac{dv(0)}{dt} = 12 \Rightarrow \frac{dv(0)}{dt} = \frac{12}{C} = 48$$

But $$\frac{dv}{dt} = \frac{d}{dt}\left[24 + \left(A_1 \cos 1.936t + A_2 \sin 1.936t\right)e^{-0.5t}\right]$$

$$= (-1.936A_1 \sin 1.936t + 1.936A_2 \cos 1.936t)e^{-0.5t}$$

$$-0.5e^{-0.5t}(A_1 \cos 1.936t + A_2 \sin 1.936t)$$

At $t = 0$, $$\frac{dv(0)}{dt} = 48 = (-0 + 1.936A_2) - 0.5(A_1 + 0) \quad (2)$$

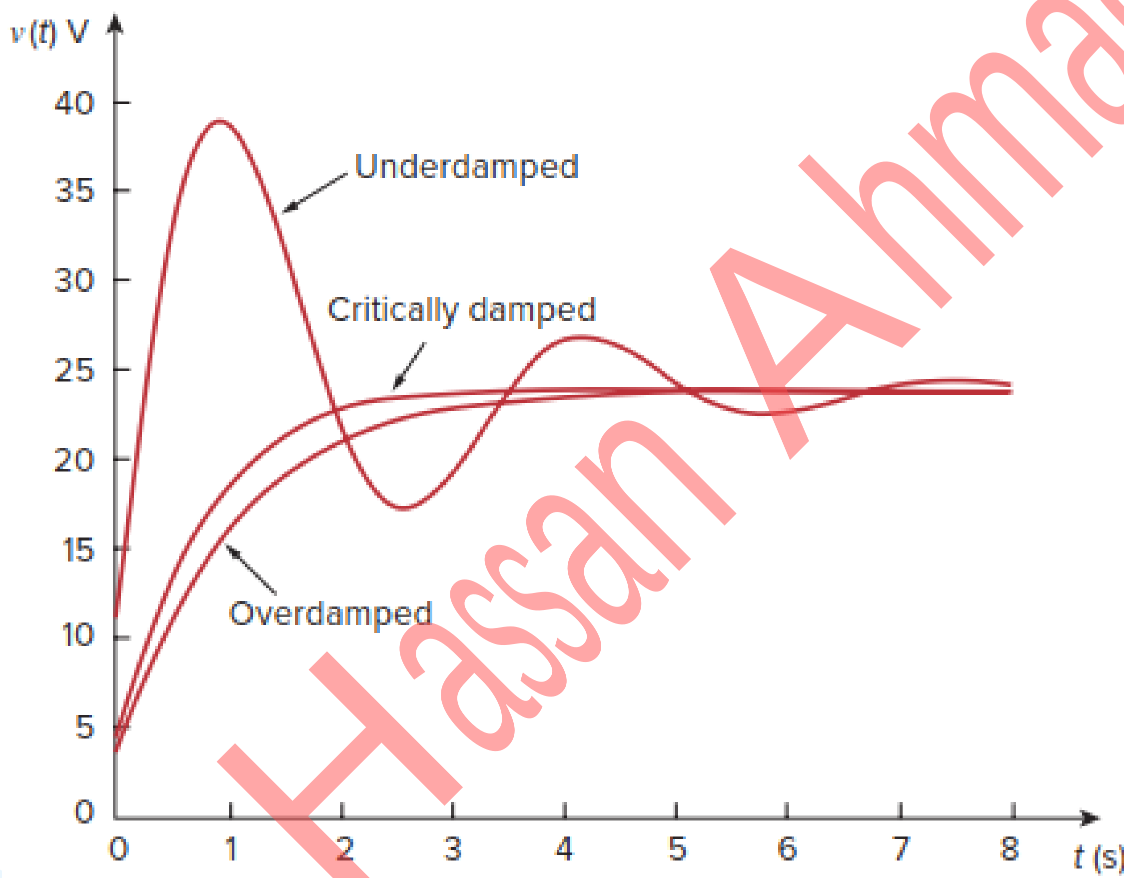
From Eqs. (1) and (2): $A_1 = -12, \quad A_2 = 21.694$

Finally, $$v(t) = 24 + \left(21.694 \sin 1.936t - 12\cos 1.936t\right)e^{-0.5t} \ \text{V}$$

The inductor current is:

$$i(t) = C\frac{dv}{dt} \Rightarrow i(t) = \left(3.1 \sin 1.936t + 12\cos 1.936t\right)e^{-0.5t} \ \text{A}$$

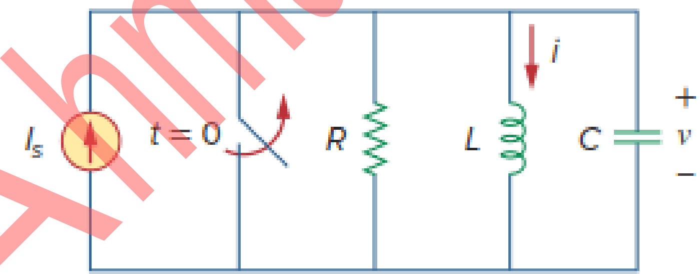
Figure plots the responses for the three cases.



From this figure, we observe that the critically damped response approaches the step input of 24 V the fastest.

8.6 Step-Response of Parallel *RLC* Circuits

- The step response is obtained by the sudden application of a dc current.
  - Applying KCL at the top node for $t > 0$,



$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s$$

with $$v = L\frac{di}{dt} \Rightarrow \frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

The complete solution consists of the *transient response* $i_t(t)$ and the *steady-state response* $i_{ss}(t)$; that is, $$i(t) = i_t(t) + i_{ss}(t)$$

The final value of the current through the inductor is the same as the source current I_s. Thus,

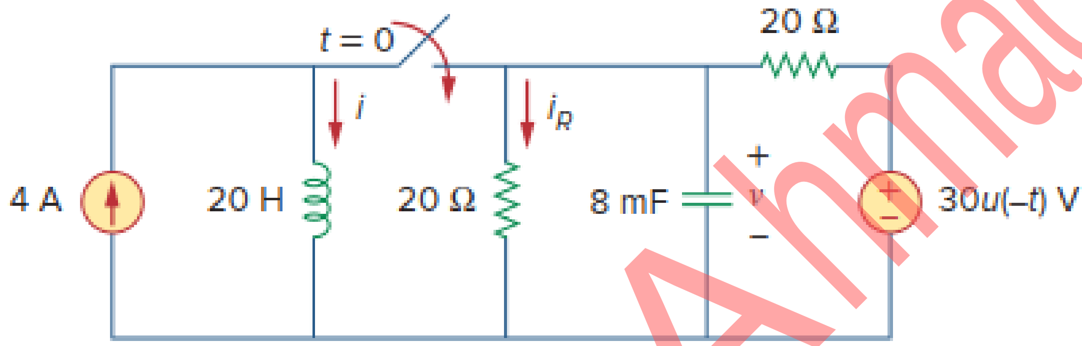
$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Ovedamped)}$$

$$i(t) = I_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)}$$

$$i(t) = I_s + \left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)e^{-\alpha t} \quad \text{(Underdamped)}$$

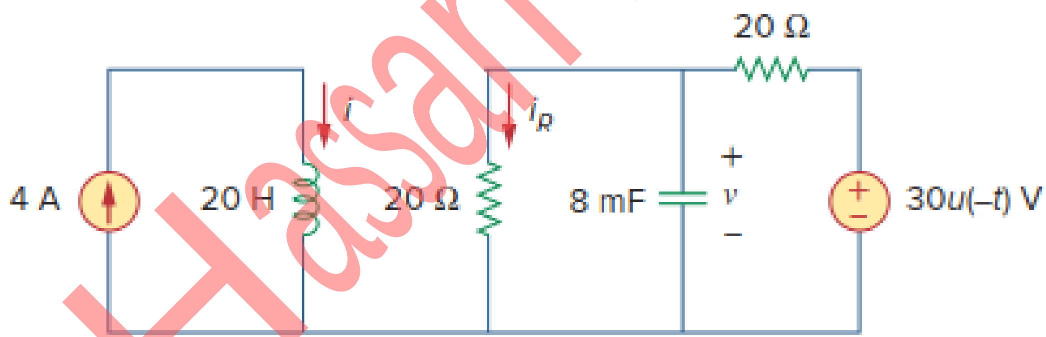
The constants A1 and A2 in each case can be determined from the initial conditions for *i* and *di/dt*.

Example 8.6. In the circuit of Fig., find $i(t)$ and $i_R(t)$ for $t > 0$.



Solution:

For $t < 0$, the switch is open, and the circuit is partitioned into two independent subcircuits, Fig.(a).



(a)

- The 4-A current flows through the inductor, so that $i(0) = 4\,\text{A}$
- $t < 0$ ➔ $30u(-t) = 30$; $t > 0$ ➔ $30u(-t) = 0$, **the voltage source is operative for $t < 0$.**

- The capacitor acts like an open circuit, and $v_{(8\text{mF})} = v_{(20\Omega)}$, Fig.(b).

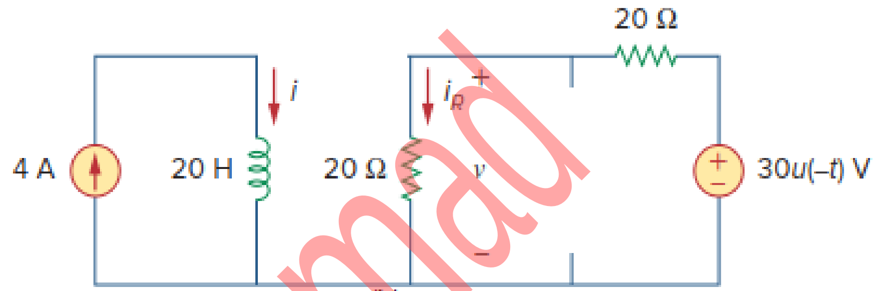
- By VDR on Fig.(b)., the **initial capacitor voltage (*t* =0)** is

$$v(0) = \frac{20}{20+20}(30) = 15\,\text{V}$$

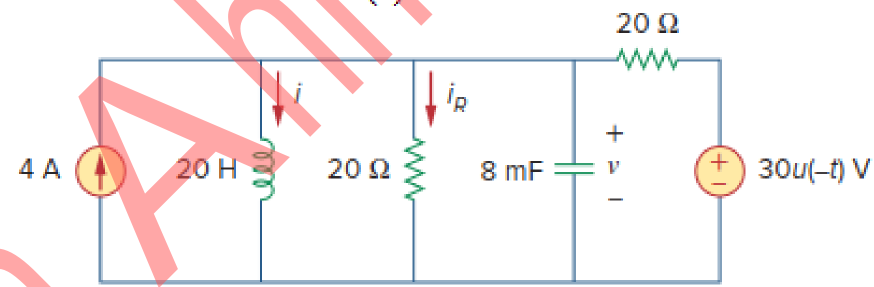
For ***t* > 0**, the switch is closed, and we have a parallel *RLC* circuit with a current source, Fig.(c).

- *t* > 0 ➔ 30*u*(− *t*) = 0, the voltage source acts like a *short-circuit*, Fig.(d).

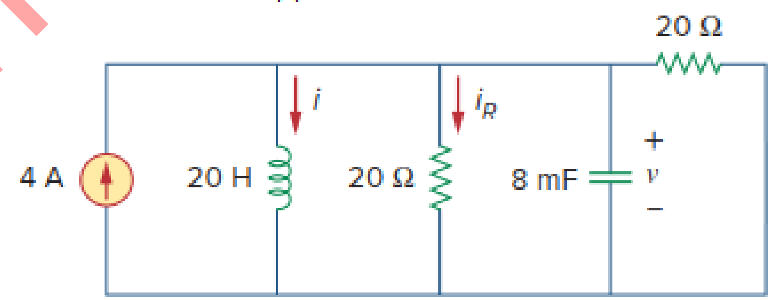
Thus, *R* = 20 ∥ 20 = 10 Ω.



(b)



(c)



(d)

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times10\times8\times10^{-3}} = 6.25, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20\times8\times10^{-3}}} = 2.5$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\ 2}} = -6.25 \pm 5.7282 \equiv s_1 = -11.978,\ s_2 = -0.5218$$

Since $\alpha > \omega_0$ we have the overdamped case. Hence,

$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} = 4 + A_1 e^{-11.978t} + A_2 e^{-0.5218t}$$

At $t = 0$, ➔ $i(0) = 4 + A_1 + A_2 = 4 \Rightarrow A_2 = -A_1 \qquad (1)$

But $\dfrac{di}{dt} = \dfrac{d}{dt}\left(4 + A_1 e^{-11.978t} + A_2 e^{-0.5218t}\right) = -11.978 A_1 e^{-11.978t} - 0.5218 A_2 e^{-0.5218t}$

so that at $t = 0$, $\dfrac{di(0)}{dt} = -11.978 A_1 - 0.5218 A_2$

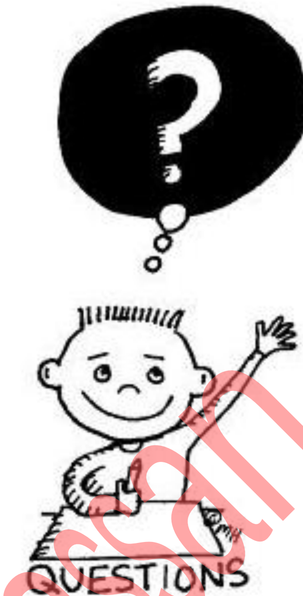
But $v(t) = L\dfrac{di(t)}{dt} \Rightarrow v(0) = L\dfrac{di(0)}{dt} = 15 \Rightarrow \dfrac{di(0)}{dt} = \dfrac{15}{L} = \dfrac{15}{20} = 0.75$

Thus, $-11.978 A_1 - 0.5218 A_2 = 0.75 \qquad (2)$

From Eqs.(1) and (2): $A_2 = 0.0655, \quad A_1 = -0.0655$

The complete solution is as $i(t) = 4 + 0.0655(e^{-0.5218t} - e^{-11.978t})$ A

$$i_R(t) = \frac{v(t)}{20} = \frac{1}{20} L \frac{di}{dt} = i(t) = 0.785 e^{-11.978t} - 0.0342 e^{-0.5218t} \text{ A}$$



The end of chapter 8